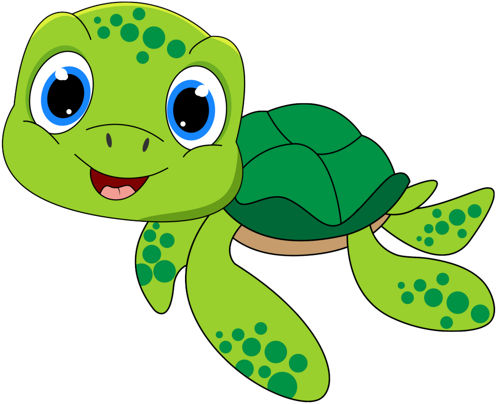
Codes for (Un)Expected Loads

Emina Soljanin, Rutgers



Acknowledgments

Collaborators:

G. Alfarano, M. Aktaș, S. Andersen, A. Johnston, G. Joshi, S. Kadhe, E. Karimi, F. Kazemi, A. Kılıç, O. Kolosev, **H. Ly Minh**, G. Matthews, C. Mayer, A. Ravagnani, A. Sprintson, G. Yadgar

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Distributed Service Model

There are n nodes providing service to multiple concurrent users, e.g., cloud edge nodes providing streaming, download, computing.

We distinguish between two functional components at each node: one for data storage and the other for service request processing.

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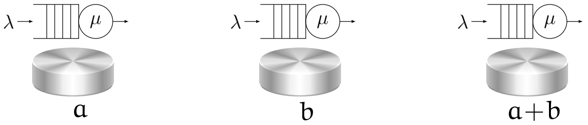
Example:
Three nodes provide data-download service to multiple concurrent users, where each user wants either data object a or data object b.



Data Storage Model

Simple Redundant Storage

- k equal size data objects are stored across n servers ($k \leqslant n$).
- Data objects are represented as elements of $\mathbb{F}_q$.
- Coded object are linear combinations of data specified by $\mathbb{F}_q^k$ vectors.
- Each server stores a single coded object (one of n).

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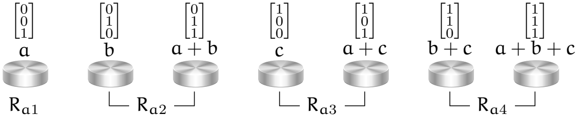
Example: Data objects a, b, and c stored across $n = 7$ nodes:

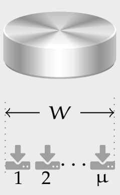


$\Longrightarrow$ a can be recovered from any of the sets $R_{a1}, R_{a2}, R_{a3}, R_{a4}$.

Data Service and Request Models

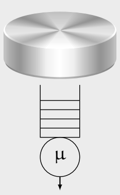
Different practical service models are mathematically equivalent.

For service, we consider the bandwidth and the queuing model:



stores one data object

Server's bandwidth W can accomodate up to μ users.



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Users queue for download.

Download is done at rate μ.

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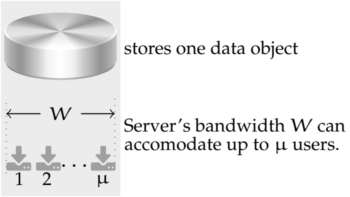
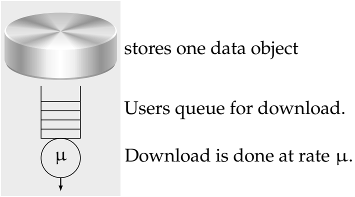




Requests for objects i, $i \in \{1, \ldots, k\}$:

- In the queuing model, requests for object i arrive at rate λ_i.
- In the bandwidth model, the number of requests for object i is λ_i

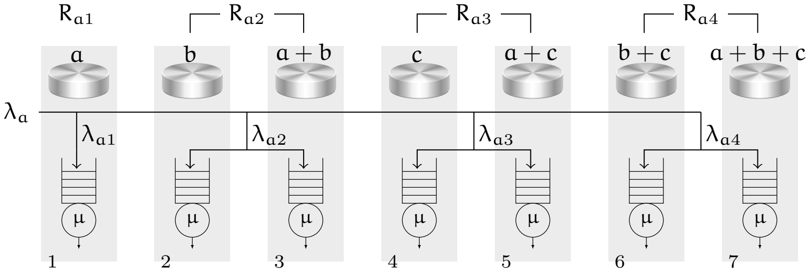
$\implies$

A server can handle multiple download requests.

Distributed Service Model – An Example

λ_a is the request rate (demand) for object a

λ_{aj} is the portion of λ_a assigned to the recovery set R_{aj}, $j \in \{1, 2, 3, 4\}$.



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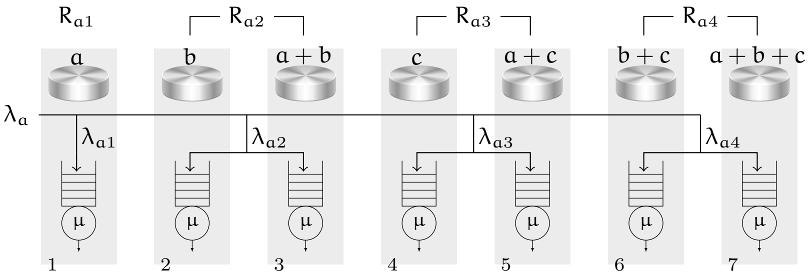

$\{\lambda_{a1}, \lambda_{a2}, \lambda_{a3}, \lambda_{a4}\}$ is a request allocation for λ_a.

Which request vectors $(\lambda_a, \lambda_b, \lambda_c)$ can be serviced by the system?

Codes with Locality and Availability in Service

What are the simultaneous recovery sets for a in the following code?

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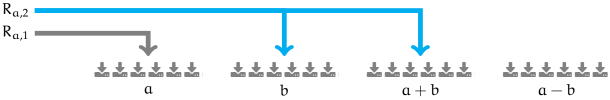

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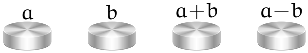

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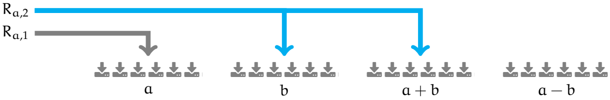
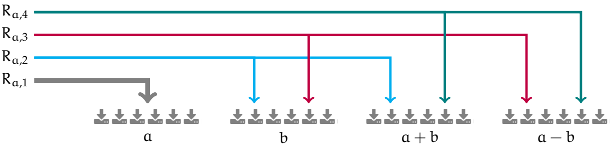


- There are four a-recovery sets in the service model:



Service Rate Region – A Polytope in $\mathbb{R}^k$

Set of vectors $(\lambda_1, \ldots, \lambda_k)$ that can be served by the system

λ_i is the request rate (demand) for object i, $i = 1, \ldots, k$.

λ_{ij} is the portion of λ_i assigned to the recovery set R_{ij}, $j = 1, \ldots, t_i$.

The request vector $(\lambda_1, \ldots, \lambda_k)$ can be serviced by the system iff there exist λ_{ij} satisfying the following constraints:

1. No server is assigned requests in excess of its service rate:

$$\sum_{i=1}^{k} \sum_{\substack{1 \leqslant j \leqslant t_i \\ \ell \in R_{ij}}} \lambda_{ij} \leqslant \mu \quad \text{for} \quad 1 \leqslant \ell \leqslant n.$$

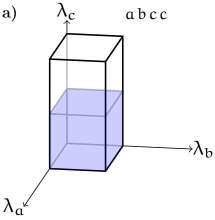
2. All objects' requests are served: $\sum_{j=1}^{t_i} \lambda_{ij} = \lambda_i$ for $1 \leqslant i \leqslant k$

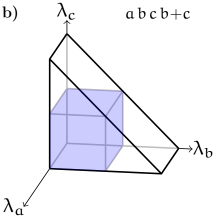
$\{\lambda_{ij} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant t_i\}$ is a request allocation for $(\lambda_1, \ldots, \lambda_k)$.

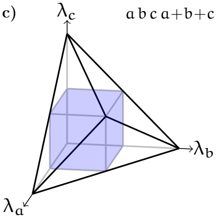
If we require that λ_{ij} be either 0 or μ, we speak of integral service rates.

Three Storage Schemes and Their Service Rates

$k = 3$ data objects stored across $n = 4$ nodes







Many (kinds of) questions are of interest.

Service Rate Region Problem(s) Formulation

System Model:

- k data objects are stored redundantly across n nodes.
- Data objects are represented as elements of some finite field.
- Each server stores a <u>linear combination</u> of data objects, i.e., a coded object of the same size (same field).
- Requests for object i, $i \in \{1, \ldots, k\}$ arrive to the system at rate λ_i.
- At each node, requests are serviced at rate $\mu = 1$.

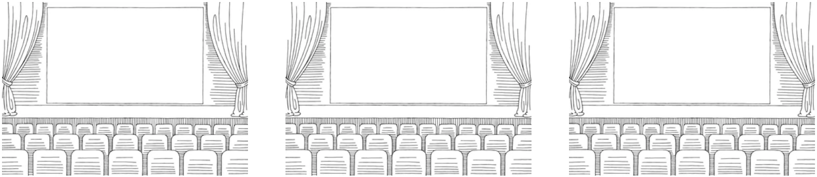
SOME OBJECTIVES:

1. <u>Determine the set of rates</u> $(\lambda_1, \ldots, \lambda_k)$  that can be supported by the system implementing some common redundancy scheme.
2. <u>Design a redundancy scheme</u> in order to maximize and/or shape the of region of supported arrival rates under some limited resources.
3. <u>Evaluate the system's performance</u> for a given stochastic model of $(\lambda_1, \ldots, \lambda_k)$ (e.g., probability of supported rates, load imbalance).

A Problem of City with Two Movies & Three Cinemas

- There are two movies a & b and three cinemas with 100 seats each.
- λ_a people want to see a and λ_b people want to see b.
- We know that the city's population is 200 $\implies \lambda_a + \lambda_b \leqslant 200$

Q1: Which movie should each cinema play?



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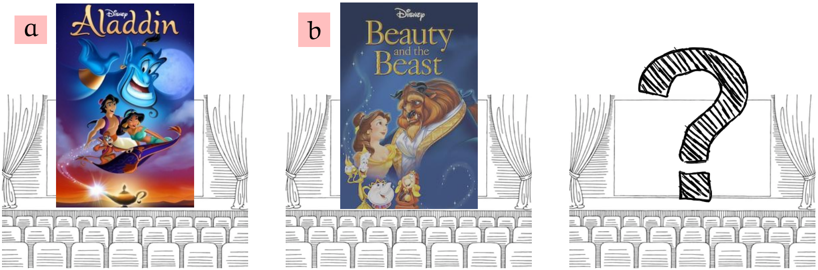

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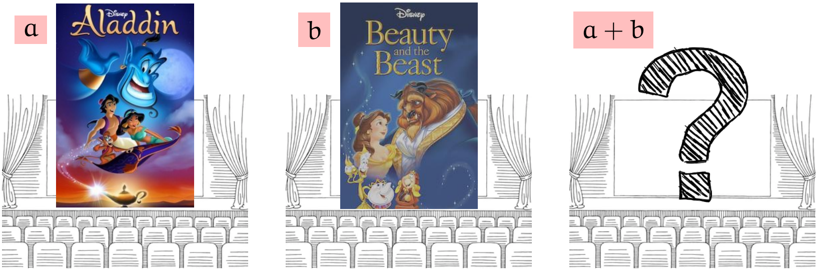

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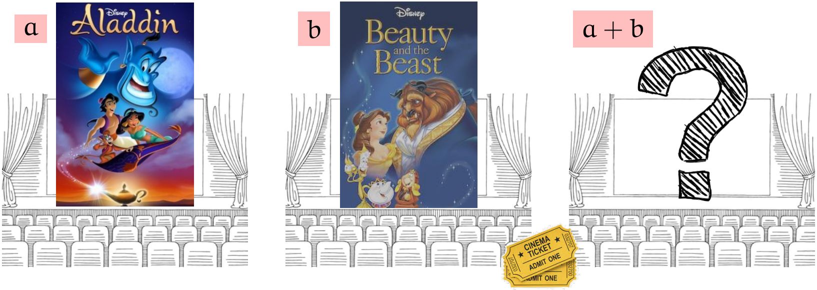

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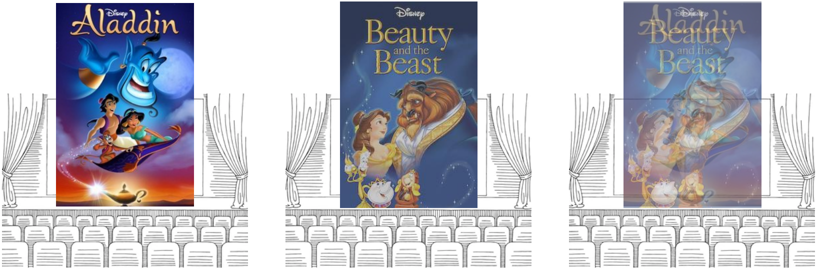
a | b | $a+b$



Q2: Can λ_a people see a and λ_b people see b as long as $\lambda_a + \lambda_b \leqslant 200$?

Q3: Which vectors (λ_a, λ_b) are possible for a given redundancy scheme?

“Matrix” G - A Collection of Storage Specifying Columns

G is a $k \times n$, $k \leqslant n$, rank-k matrix &

 Columns of G are a multi-set V of points in $\mathbb{PG}(k-1, q)$.

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Example #1 – the [7, 3] binary Simplex code

Matrix $G = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$ encodes data $[c \;\; b \;\; a]$ as follows:

$$[c \;\; b \;\; a] \cdot G = [a \;\; b \;\; a{+}b \;\; c \;\; a{+}c \;\; b{+}c \;\; a{+}b{+}c]$$

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Example #2 – two $[4,2]$ MDS codes (α is a primitive in $\mathbb{F}_q$, $q > 4$.)

$$G_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{bmatrix}$$

These storage schemes provide different data access performance.

Minimal Recovery Sets

Subset R of columns in G is a minimal recovery set of data object a if

- $a \in \text{span}(R)$
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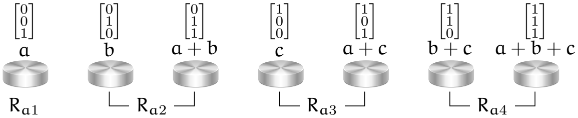

R_{a1}, R_{a2}, R_{a3}, and R_{a4} are the recovery sets of size one and two of a in

$$G = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$


There is a minimal recovery set for a of size three!

Minimal Recovery Sets

Subset R of columns in G is a <u>recovery set of e_j</u> if

- $e_j \in \mathrm{span}(R)$
- $S \subset R \implies e_j \notin \mathrm{span}(S)$

<u>Example:</u>

Let $G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{bmatrix}$. Then

- the (size one and two) recovery sets of $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \; \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \; \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\} \; \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\}$$

- the (size one and two) recovery sets of $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \; \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \; \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\} \; \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\}$$

 The recovery sets do not have to be disjoint (as for LRCs).

The Recovery Graph Γ_G of a $2 \times n$ Matrix G

and other matrices with only size-2 recovery sets

NODES:

Γ_G has n nodes corresponding to the columns of G, and
i additional nodes $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_j$ for systematic columns $j = 1, \ldots, i$.

EDGES:

If two nodes correspond to a recovery set of e_j,
they are connected by an edge which is given label e_j.

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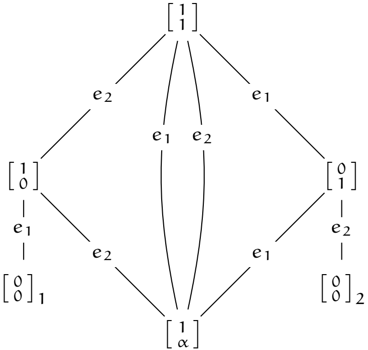
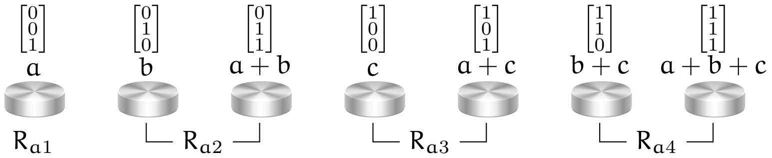
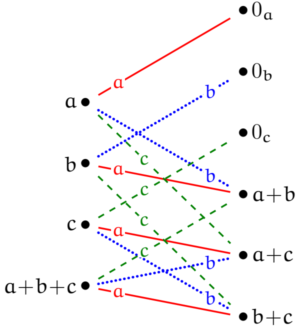
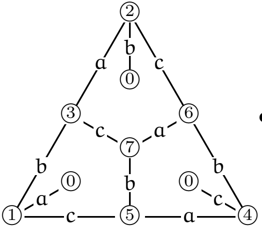


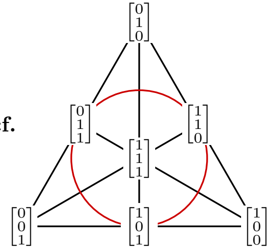
Recovery Graphs of Binary Simplex Codes







cf.



Fractional Matching & Service Rates on Recovery Graphs

A fractional matching in (V, E) is a vector $\boldsymbol{w} \in \mathbb{R}^{|E|}$ whose components (weights) w_ϵ, $\epsilon \in E$, are non-negative and $\sum_{\epsilon \ni v} w_\epsilon \leqslant 1$ for each $v \in V$.

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We define $\lambda_j^{\boldsymbol{w}}$, the service rate for e_j in matching $\boldsymbol{w}$ as the sum of weights w_ϵ of all e_j-labeled edges $\epsilon \in E$. We say that $\boldsymbol{w}$ yields $\lambda_j^{\boldsymbol{w}}$.

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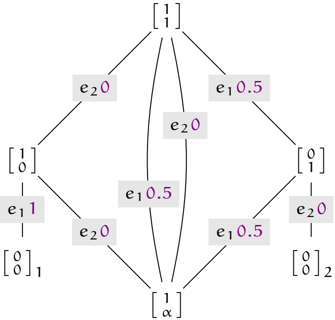
A matching with service rates $\lambda_1 = 2.5$ and $\lambda_2 = 0$.

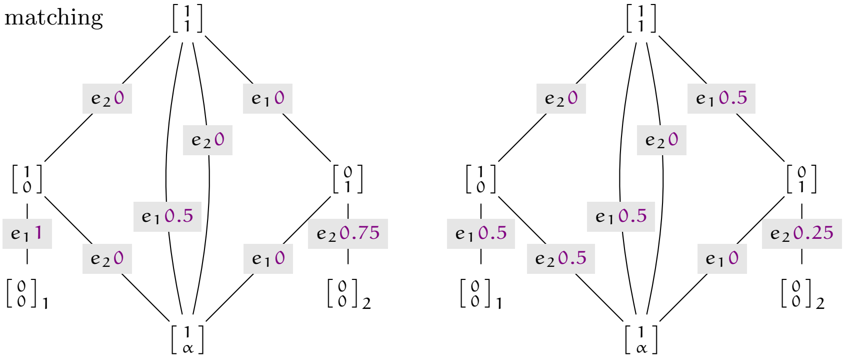
$$G_2(4,2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{bmatrix} \longrightarrow \Gamma_2(4,2)$$



A Service Vector and its Matchings

Two matchings for $(1.5, 0.75)$ service vector in $\Gamma_2(4,2)$:

greedy matching



Fractional Matching and Service Polytopes of $\Gamma_G = (V, E)$

<u>The set of all fractional matchings</u> in $\Gamma_G = (V, E)$ is a polytope in $\mathbb{R}^{|E|}$, called the **fractional matching polytope** and denoted by $\mathrm{FMP}(\Gamma_G)$.

<u>The set of all service vectors</u> in $\Gamma_G = (V, E)$ is a polytope in $\mathbb{R}^k$. We call it the **service rate region**, and denote by $\mathcal{R}(\Gamma_G)$.

$\implies$ $\mathcal{R}(\Gamma_G)$ is the image of $\mathrm{FMP}(\Gamma_G)$ under a linear map from $\mathbb{R}^{|E|}$ to $\mathbb{R}^k$.

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The service rate region problem:

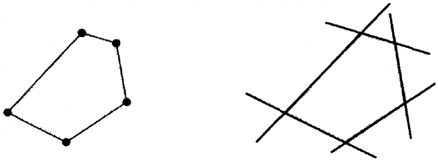
$$\boxed{\text{Find } \mathcal{R}(\Gamma_G) = \Big\{\boldsymbol{\lambda}^{\boldsymbol{w}} \in \mathbb{R}^k : \boldsymbol{w} \in \mathrm{FMP}(\Gamma_G), \lambda_i^{\boldsymbol{w}} = \sum_{\epsilon \text{ labeled by } e_i} w_\epsilon \Big\}.}$$

What does it mean to characterize a polytope?

We need to specify points or hyperplanes!

Convex Polytope in $\mathbb{R}^{\mathrm{d}}$:

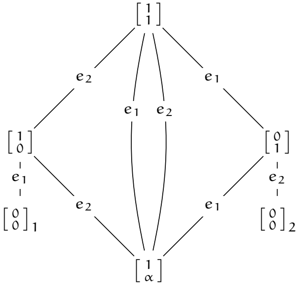
- convex hull of a finite set of points OR
- intersection of finitely many closed half spaces that is bounded

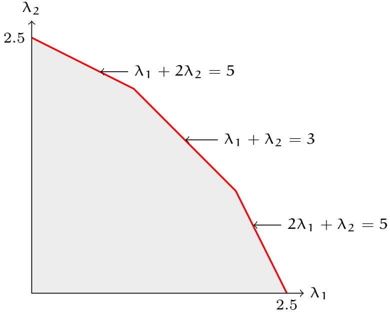


Ziegler's Lectures on Polytopes, Springer, GTM 152, revised first edition.

A Service Rate Polytope Example

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{bmatrix} \longrightarrow \Gamma_G$$

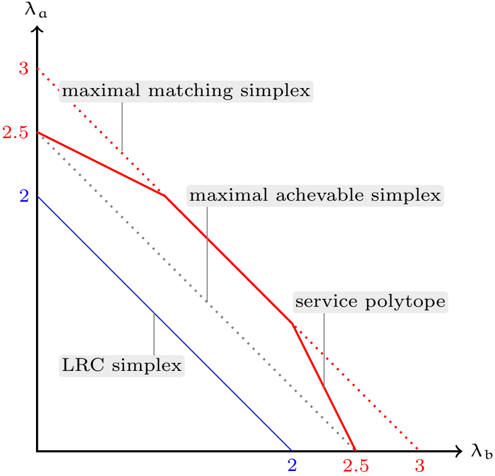




Some bounds on $\mathcal{R}(\Gamma_G)$ follow from the fractional matching number of Γ_G.

Two Bounding Simplexes (and the LRC Simplex)

Bounds on $\sum_{i=1}^{k} \lambda_i$ for any vertex $(\lambda_1 \ldots, \lambda_k)$ of the service region



We often compare service rate regions based on their bounding simplexes.

The Maximal Matching Simplex

Fractional Matching and Vertex Cover Numbers

For matching $\boldsymbol{w}$, we have $\displaystyle\sum_{i=1}^{k} \lambda_i^{\boldsymbol{w}} = \sum_{\epsilon \in E} w_\epsilon$, $\longleftarrow$ the size of $\boldsymbol{w}$.

$\Longrightarrow$

For all $\boldsymbol{\lambda} \in \mathcal{R}(\Gamma_G)$, we have $\displaystyle\sum_{i=1}^{k} \lambda_i \leqslant \max_{\boldsymbol{w} \in FMP(\Gamma_G)} \sum_{\epsilon \in E} w_\epsilon = \nu^*(\Gamma_G)$

$\nu^*(\Gamma_G)$ is the fractional matching number of Γ_G.

$FMP(\Gamma_G) = \{\boldsymbol{w} \in \mathbb{R}^{|E|} : A\boldsymbol{w} \leqslant \mathbf{1},\ \boldsymbol{w} \geqslant 0\} \Longrightarrow$ finding $\nu^*(\Gamma_G)$ is an LP problem.

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Its dual finds the fractional vertex cover number $\tau^*(\Gamma_G) = \nu^*(\Gamma_G)$:

$$\tau^*(\Gamma_G) = \min \sum_{v \in V} \omega_v \text{ s.t. } A^T \boldsymbol{\omega} \geqslant \mathbf{1},\ \boldsymbol{\omega} \geqslant \mathbf{0}$$

A fractional vertex cover in (V, E) is a vector $\boldsymbol{\omega} \in \mathbb{R}^{|V|}$ whose components (weights) ω_v, $v \in V$, are non-negative and $\sum_{v \in \epsilon} \omega_v \geqslant 1$ for each $\epsilon \in E$.

The Maximal Matching Simplex

Axes Intercept Points of $\mathcal{R}(\Gamma_G)$

Two easy-to-prove observations:

1. Let $\lambda_j^{\max} = \max_{\boldsymbol{\lambda} \in \mathcal{R}(\Gamma_G)} \lambda_j$. Then $\lambda_j^{\max} e_j$ is a vertex of $\mathcal{R}(\Gamma_G)$.
2. Let Γ_G^j be the sub-graph of Γ_G induced by e_j-edges. Then $\lambda_j^{\max}$ is equals to the **matching number** of Γ_G^j.

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The convex hull of points 0 and $\lambda_j^{\max} e_j$ is a k-simplex within $\mathcal{R}(\Gamma_G)$.

Bounding Simplexes for $G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{bmatrix}$

<u>maximal matching simplex</u>:

intersection of half spaces

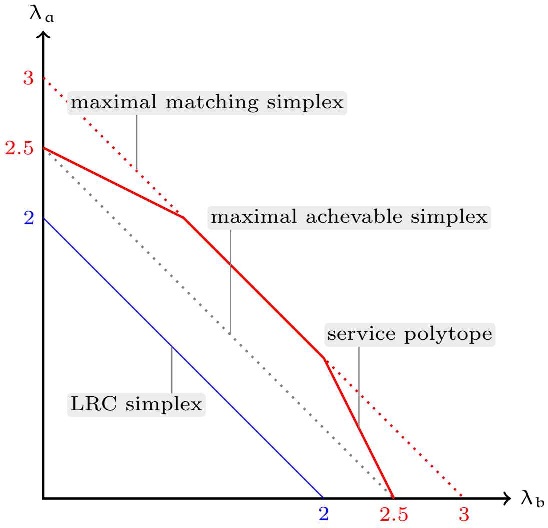
$\sum_{i=1}^{k} \lambda_i \leqslant \nu^*(\Gamma_G)$ & $\boldsymbol{\lambda} \geqslant 0$.

(Some points are achievable!)

<u>maximal achievable simplex:</u>

convex hull of points

$\lambda_i^{\max} e_i, \; i = 1, \ldots, k, \;$ & $\boldsymbol{\lambda} = 0$.



For binary simplex codes, these simplexes coincide.

Binary Simplex Codes and their Recovery Graphs

aka Hadamard Codes in CS literature

G_k consist of all distinct nonzero vectors of $\mathbb{F}_2^k$.
$\implies$ Γ_k nodes are labeled by k-bit stings with even and odd weight.

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Lemma: Structure of the recovery graph Γ_k:

1. Γ_k is bipartite. Edges connect odd-weight with odd-weight nodes.
2. Each odd-weight node of Γ_k has degree k.
3. The 2^{k-1} odd-weight nodes of form a minimum vertex cover of Γ_k.

Service Rate Region of $[2^k - 1, k]$ Binary Simplex Codes

Theorem: A simplex in $\mathbb{R}^k$!

The service region of the $[2^k - 1, k]$ binary Simplex code is defined by

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leqslant 2^{k-1}, \quad \lambda_i \geqslant 0, \ i = 1, \ldots, k.$$

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Proof Sketch for the Achevability:

Fractional matching on the recovery graph that assigns weight $\lambda_i / 2^{k-1}$ to each e_i-labeled edge gives $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 2^{k-1}$.

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Proof Sketch for the Converse:

For bipartite graphs, the size of the minimum vertex cover (here 2^{k-1}) is equal to the (fractional) matching number.

A Class of MDS Matrices $G_i(n,k)$, $i = 0,1,2,\dots,k$

α is a primitive in $\mathbb{F}_q$, $q > n$

$$G_0(n,k) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{i-1} & \dots & \alpha^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \alpha^{k-1} & \alpha^{2(k-1)} & \dots & \alpha^{(i-1)(k-1)} & \dots & \alpha^{(n-1)(k-1)} \end{bmatrix},$$

$$G_i(n,k) = \left[\begin{array}{cccc|ccc} 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 0 & 1 & \dots & 0 & \alpha^i & \dots & \alpha^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \alpha^{(i-1)i} & \dots & \alpha^{(i-1)(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad 0}_{} & & \dots & 0 & \alpha^{(k-1)i} & \dots & \alpha^{(k-1)(n-1)} \end{array}\right] \text{ for } i = 1,\dots,k.$$

$$\underbrace{\qquad\qquad\qquad}_{i \text{ columns of } I_k} \quad \underbrace{\qquad\qquad\qquad\qquad}_{n-i \text{ columns of } G_0(n,k)}$$

We denote the j-th column of I_k by e_j and call it systematic, $j = 1,\dots,k$.

Quasi-Uniform Recovery Hypergraphs

A hypergraph is a pair (V, E), where

- V is a finite set, the set of of vertices &
- E is a multiset of subsets of V called edges.

Uniform and Quasi-uniform Hypergraphs

We say that a hypergraph is

- k-uniform if each of its edges has size k
- (k, m)-quasi-uniform if each of its edges has either size k or m.

$\implies$ Graphs are 2-uniform hypergraphs.

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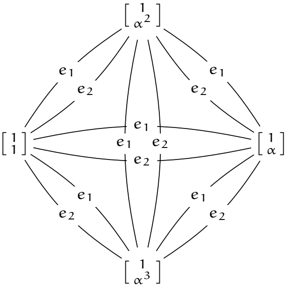
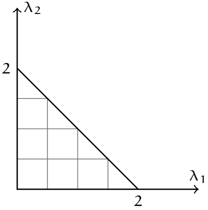
$\Longrightarrow$ Graphs are 2-uniform hypergraphs.

Q: What about recovery graphs $\Gamma_i(n, k)$ of $G_i(n, k)$, $i = 0, 1, 2, \ldots, k$?
A: They are $(k, 2)$-quasi-uniform and $\Gamma_0(n, k)$ is k-uniform.
The systematic edges have size 2; all other edges have size k.

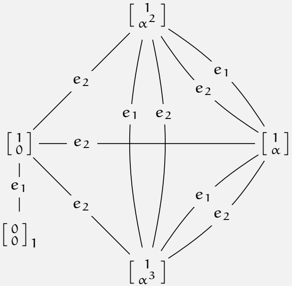
$[4, 2]$ MDS Matrices, Recovery Graphs, and Rate Regions

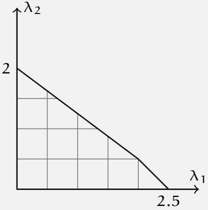
$$G_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 \end{bmatrix}$$



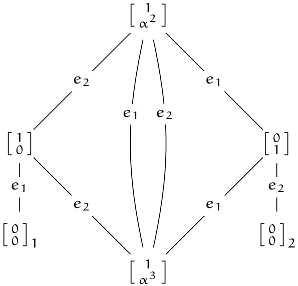


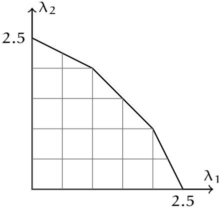
$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha & \alpha^2 & \alpha^3 \end{bmatrix}$$





$$G_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \alpha^2 & \alpha^3 \end{bmatrix}$$





An Inclusion Theorem for $\mathcal{R}_i(n, k), i = 0, 1, \ldots, k$

The proof follows from characterizing the maximal achievable and matching simplexes

For any k and $n > k$, we have

$$\mathcal{R}_0(n, k) \subset \mathcal{R}_1(n, k) \subset \mathcal{R}_2(n, k) \subset \cdots \subset \mathcal{R}_{k-1}(n, k) \subset \mathcal{R}_k(n, k)$$

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Can we completely characterize these polytopes?

Yes, we can, but with much more work.

Perfect Matching, Matching Bound, and Greedy Matchings

TOOLS: New & Old

A fractional matching in (V, E) is a vector $\boldsymbol{w} \in \mathbb{R}^{|E|}$ whose components (weights) w_ϵ, $\epsilon \in E$, are non-negative and $\sum_{\epsilon \ni v} w_\epsilon \leqslant 1$ for each $v \in V$.
$\implies$ the matching bound

$$|V| \geqslant \sum_{v \in V} \sum_{\epsilon \ni v} w_\epsilon = \sum_{\epsilon \in E} w_\epsilon |\epsilon|.$$

A matching that saturates the bound is called perfect.

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A matching that saturates the bound is called perfect.

Greedy Matching Theorem

Let $\boldsymbol{\lambda}$ be a point in $\mathcal{R}_i(n, k)$, i.e., $\exists \boldsymbol{w} \in \mathrm{FMP}(\Gamma_i(n, k))$ s.t. $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\boldsymbol{w}}$.

Then there exist a greedy matching $\boldsymbol{\sigma} \in \mathrm{FMP}(\Gamma_i(n, k))$ s.t. $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\boldsymbol{\sigma}}$ and the weight of the j-th systematic edge $\sigma_j = \min\{1, \lambda_j\}$, for all $j \leqslant i$.

$\mathcal{R}_i(n,k)$ for $n \geqslant 2k$

Slicing $\Gamma_i(n,k)$, $n - i \geqslant k$, into k-Uniform Subgraphs

Consider $\boldsymbol{\lambda} = (\underbrace{\lambda_1, \ldots, \lambda_{i_A}}_{\geqslant 1}, \underbrace{\lambda_{i_A+1}, \ldots, \lambda_i, \ldots, \lambda_k}_{<1}) \in \mathcal{R}_i(n,k)$

+ matching constraint at nodes, greedy matching, $\Gamma_i(n,k)$ structure.

$\implies$ There is a perfect matching that saturates the matching bound

$$k \cdot \sum_{j=1}^{i_A} (\lambda_j - 1) + \sum_{j=i_A+1}^{i} \lambda_j + k \cdot \sum_{j=i+1}^{k} \lambda_j \leqslant n - i_A$$

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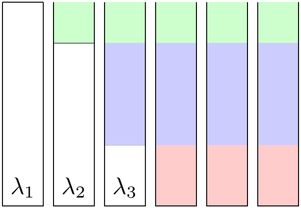
Example: A perfect matching on $\Gamma_3(6,3)$ with $i_A = 1$, λ_2=0.8, $\lambda_3 = 0.3$.



$\mathcal{R}_i(n,k)$ for $n \geqslant 2k$

Observe that when $n \geqslant 2k$, then $n - i \geqslant k$ for all $i = 0, 1, 2, \ldots, k$.

Consider $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathcal{R}_i(n,k)$ and a partition of $\{1, \ldots, k\}$ into sets A, B, C s.t. if $j > i$, $j \in C$; if $j \leqslant i$, $j \in A$ if $\lambda_j \geqslant 1$, otherwise $j \in B$.

$\Longrightarrow$

$\mathcal{R}_i(n,k)$ is the intersection for the following half spaces:

$\boldsymbol{\lambda} \geqslant \mathbf{0}$ and

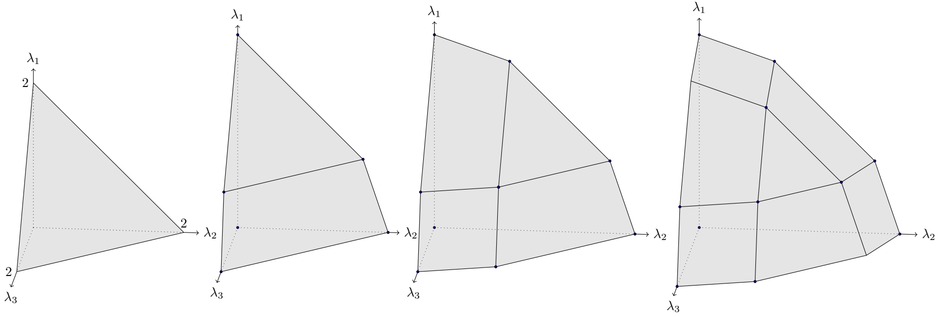
$$k \cdot \sum_{j \in A} (\lambda_j - 1) + \sum_{j \in B} \lambda_j + k \cdot \sum_{j \in C} \lambda_j \leqslant n - |A|,$$

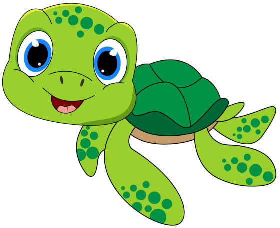
for all partitions $A \cup B = \{1, \ldots, i\}$ s.t. $\lambda_j \geqslant 1$ for $j \in A$ and $\lambda_j < 1$ for $j \in B$, and $C = \{i+1, \ldots, k\}$. If $i = k$, then $A = \emptyset$ does not give an active constraint.

$\Longrightarrow$

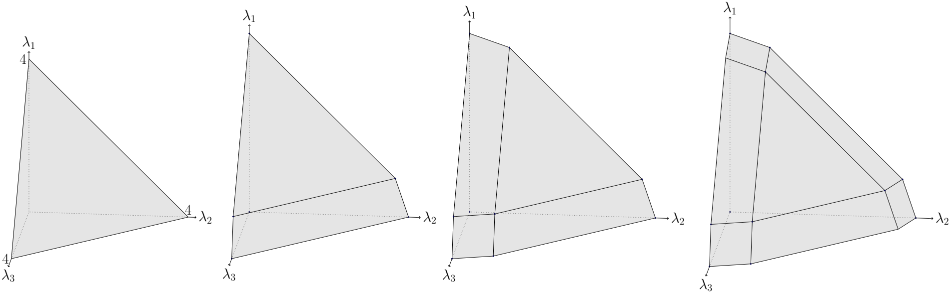
There are $k + 2^i$ hyperplanes for $i = 0, \ldots, k-1$, and $k + 2^k - 1$ for $i = k$.

$\mathcal{R}_i(6,3)$, $i = 0, 1, 2, 3$





$\mathcal{R}_i(12, 3)$, $i = 0, 1, 2, 3$



$\mathcal{R}_i(n,k)$ for $n - i < k$

$\Longrightarrow$ There may not be a matching that saturates the matching bound

$$k \cdot \sum_{j=1}^{i_A} (\lambda_j - 1) + \sum_{j=i_A+1}^{i} \lambda_j + k \cdot \sum_{j=i+1}^{k} \lambda_j \leqslant n - i_A$$

What about the vertex cover bound $\sum_{j=1}^{k} \lambda_j \leqslant i$?

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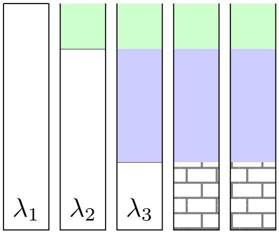
What about the vertex cover bound $\sum_{j=1}^{k} \lambda_j \leqslant i$?

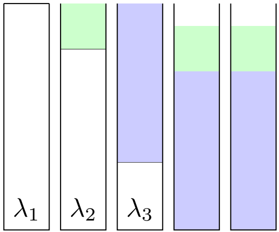
Example: Two matchings on $\Gamma_3(5, 3)$ with $i_A = 1$, λ_2=0.8, $\lambda_3 = 0.3$.

size $1 + \frac{4}{3} + \frac{2}{3}\lambda_2$



hits a brick wall

size 3



saturates the vertex cover bound

The Vertex Cover Bound on $\mathcal{R}_k(n, k)$ for $n < 2k$

Multiple points in $\mathcal{R}_k(n, k)$ saturate the bound when $n = 2k - 1$

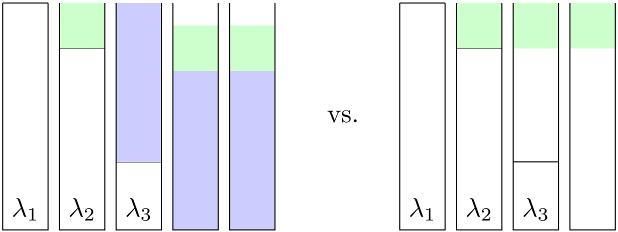
$\boldsymbol{\lambda} = (1, \ldots, 1)$ is the only point reaching the bound when $n < 2k - 1$.

The Vertex Cover Bound on $\mathcal{R}_k(n,k)$ for $n < 2k$

Multiple points in $\mathcal{R}_k(n,k)$ saturate the bound when $n = 2k-1$

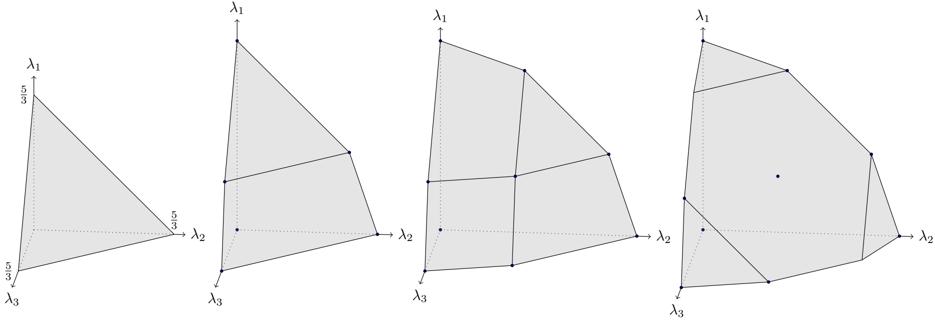
$\boldsymbol{\lambda} = (1,\ldots,1)$ is the only point reaching the bound when $n < 2k-1$.

Example: $\Gamma_3(5,3)$ vs. $\Gamma_3(4,3)$ when $i_A = 1$, λ_2=0.8, $\lambda_3 = 0.3$.



$\mathcal{R}_i(5,3)$, $i = 0,1,2,3$

and some remarks on $\mathcal{R}_i(2k-1,k)$

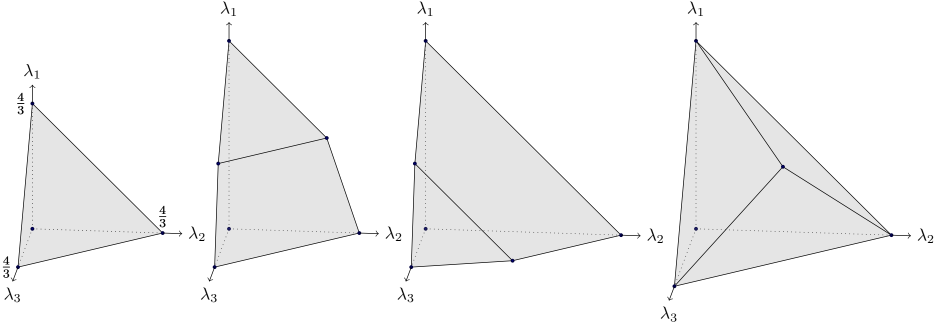


In $\mathcal{R}_i(2k-1,k)$,

- if $i < k$, then $n - i \geqslant k$ for which we know $n + 2^i$ matching bounds.
- if $i = k$, multiple points achieve the vertex cover bound.

$\mathcal{R}_i(4,3)$, $i = 0, 1, 2, 3$

and some remarks on $\mathcal{R}_i(n,k)$ for $n < 2k-1$



In $\mathcal{R}_i(n,k)$ for $n < 2k-1$,

- if $n - i \geqslant k$ and we have $n + 2^i$ matching bounds.
- if $i = k$, a single point achieves the vertex cover bound.

Service Rate Region Problem(s) Formulation

System Model:

- k data objects are stored redundantly across n nodes.
- Data objects are represented as elements of some finite field.
- Each server stores a linear combination of data objects, i.e., a coded object of the same size (same field).
- Requests for object i, $i \in \{1, \ldots, k\}$ arrive to the system at rate λ_i.
- At each node, requests are serviced at rate $\mu = 1$.

SOME OBJECTIVES:

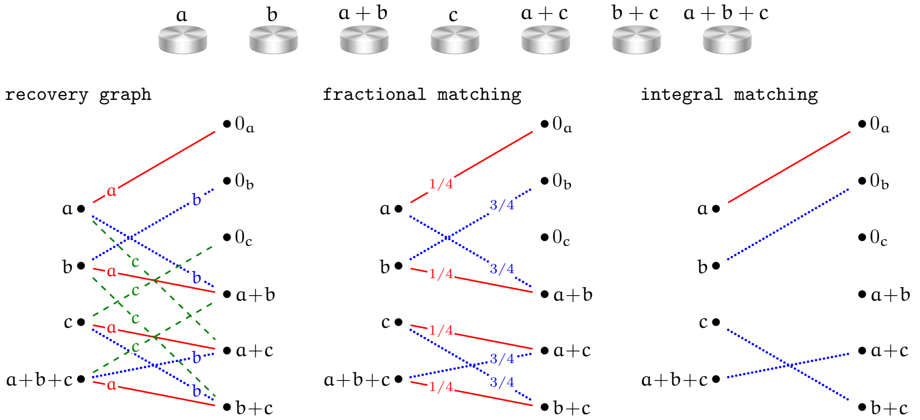
1. Determine the set of rates $(\lambda_1, \ldots, \lambda_k)$  that can be supported by the system implementing some common redundancy scheme.
2. Design a redundancy scheme in order to maximize and/or shape the of region of supported arrival rates under some limited resources.
3. Evaluate the system's performance for a given stochastic model of $(\lambda_1, \ldots, \lambda_k)$ (e.g., probability of supported rates, load imbalance).

Asynchronism

Asynchronous Service Rate Region

Asynchronous Batch Codes by Riet, Skachek, and Thomas

Consider the $(7,3)$ simplex code and two ways to satisfy demand $(1,3,0)$:



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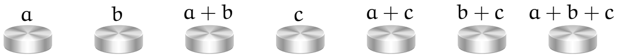
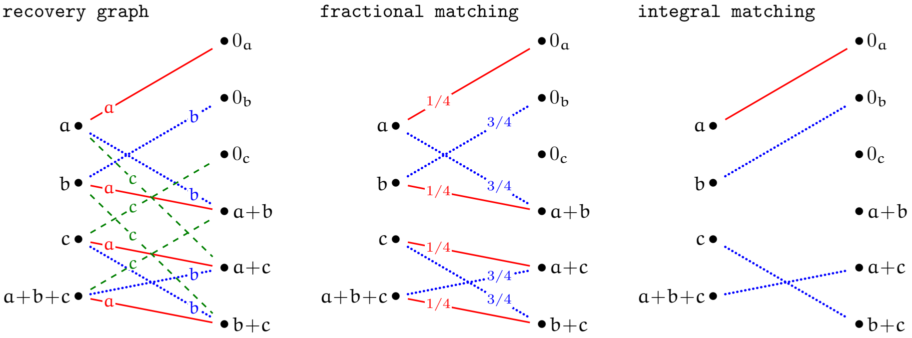
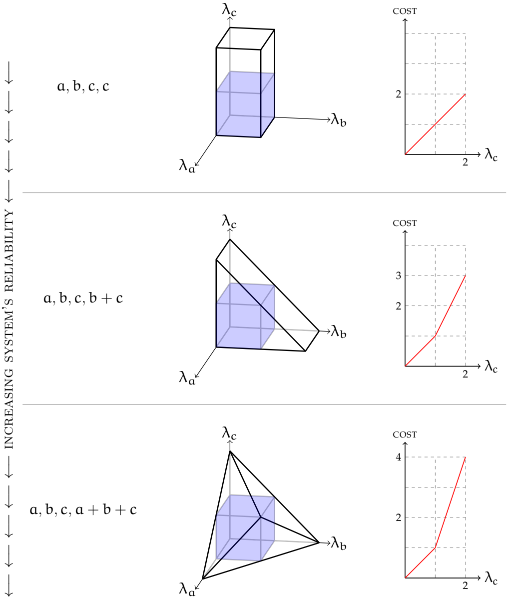




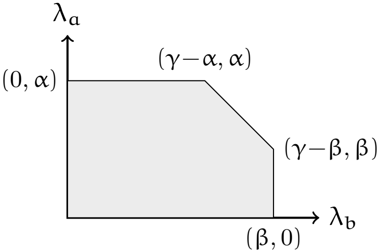
Q: If some users leave the system, can others use the freed resources?

Benefits and Costs of Adding Server(s)



Covering a Region with Minimal Storage

We need to serve requests in the region $\lambda_a \leqslant \alpha$, $\lambda_b \leqslant \beta$, $\lambda_a + \lambda_b \leqslant \gamma$.



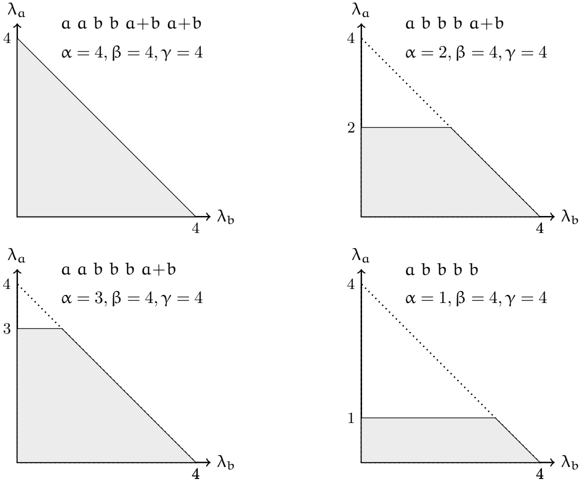
The columns of the generator matrix can only be $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.

$$\underbrace{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\ldots\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}_{n_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}}\quad\underbrace{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ldots\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}_{n_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}}\quad\underbrace{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ldots\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}_{n_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}}$$

Find $n_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}, n_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}, n_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}$ that minimize $n = n_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} + n_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} + n_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}$.

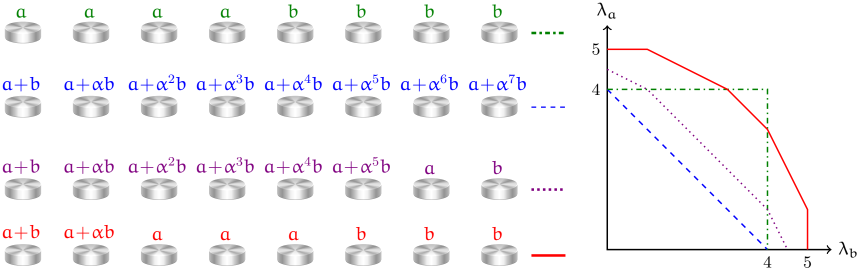
Covering a Region with Minimal Storage – Examples

What is the minimal number of servers and the redundancy scheme that satisfy the demand described by $\lambda_a \leqslant \alpha$, $\lambda_b \leqslant \beta$, $\lambda_a + \lambda_b \leqslant \gamma$?



Maximizing Service Rate Region with Fixed Resources

How should we store k objects on n servers?



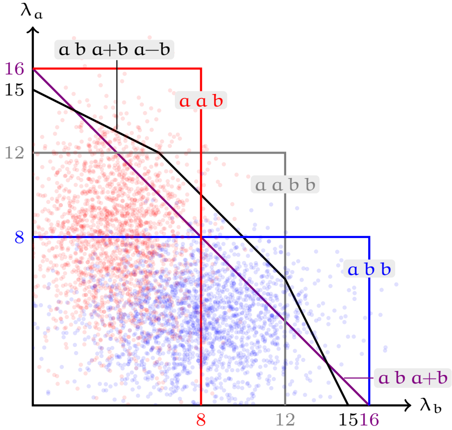
- Combining coding and replication is beneficial in multiple ways.
- Service rate region depends on the generator matrix of the code.

Covered Requests, Server Utilization, Load (Im)balance

Requests: $\lambda_a \sim \mathcal{N}^+(4,4)$ and $\lambda_b \sim \mathcal{N}^+(8,8)$ and vice versa.
Two systems with equal total service bandwidth, storing $k = 2$ objects.

System 1: $n = 3$ with $\mu = 8$ with codes
$[a, a, b]$ $[a, b, b]$ $[a, b, a{+}b]$

System 2: $n = 4$ with $\mu = 6$ with codes
$[a, a, b, b]$ $[a, b, a{+}b, a{-}b]$



Request coverage: 0.7366 for $[a, a, b]$ & $[a, b, b]$, 0.8727 for $[a, b, a{+}b]$
0.9211 for $[a, a, b, b]$, and 0.9434 $[a, b, a{+}b, a{-}b]$.

Codes for (Un)Expected Loads

New applications create new performance metrics for codes, and thus the needs for new coding schemes to be designed.